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On Simultaneous Best Approximations in $C^1[a, b]$

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INTRODUCTION

Brosowski [2] has proved the following:

THEOREM. *Let $V_1 \subset V_2$ be alternation systems of $C[a, b]$, $v_1 \in V_1$ and $v_2 \in V_2$. In order that there exists an element $f \in C[a, b]$ such that $v_i \in P_{V_i}(f)$, $i = 1, 2$, it is necessary and sufficient that $v_1 - v_2$ is either zero or changes sign at atleast $d(v_1)$ points of $[a, b]$. If V_1 and V_2 are contained in $C^2[a, b]$, then we can choose f to be a polynomial.*

For the case of polynomials, Rivlin [7] posed the following:

PROBLEM. *Characterize those n-tuples of algebraic polynomials $\{p_0, p_1, \ldots, p_{n-1}\}$ with degree of $p_i = i$, $i = 0, 1, \ldots, n-1$, for which there exists an f in $C[a, b]$ such that the polynomial of best approximation of degree i to f is p_i, $i = 0, 1, \ldots, n-1$.*

He has shown that for this to be true it is necessary for each pair i, j with $0 \leqslant i < j \leqslant n-1$ that the polynomial $p_i - p_j$ is either zero or changes sign at atleast $i+1$ points of $[a, b]$. Deutsch, Morris, and Singer [5] have shown that in the case $n = 2$, the condition is also sufficient. Sprecher [8] has extended this result to the case of two polynomials of arbitrary degrees and in [9] has given a solution to the above problem for $n = 3$. Hegering [6] and Subrahmanya [10] have considered general Chebyshev systems of $C[a, b]$. Subrahmanya [11] has given a solution to the Rivlin problem for a general n. Brosowski and Subrahmanya [3] have considered $C[T]$, where T is a compact Hausdorff space and have characterized an infinite set of elements for which there exists an $f \in C[T]$ with this set of elements as best approximations from arbitrary subsets which they assume suns only in the necessity part. Hegering [6] has also proved the following:

Let $V_1 \subset V_2$ be Haar subspaces of $C[a, b]$ and $v_i \in V_i$, $i = 1, 2$. Let

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$v_1 \in C^h[a, b]$ and $v_2 \in C^l[a, b]$, $h, l \in \mathbb{N}$ ($h, l = \infty$ allowed). Under the necessity condition of Rivlin there exists an $f \in C^{\min(h,l)}[a, b]$ such that $v_i \in P_{V_i}(f)$, $i = 1, 2$.

In this paper we generalize this result to alternation systems. Our proof is simple and straight forward.

We now give a brief review of some notation. Let $C[a, b]$ denote the set of all continuous real-valued functions defined on $[a, b]$. $C[a, b]$ is equipped with the norm given by

$$\|f\| := \sup_{t \in [a,b]} |f(t)|.$$

Let $V \subset C[a, b]$ and $f \in C[a, b]$. An element $v_0 \in V$ is said to be a best approximation to f in V, if

$$\|f - v_0\| = \inf_{v \in V} \|f - v\| =: E_V(f).$$

We denote by $P_V(f)$ the set of all best approximations to f from V, that is,

$$P_V(f) = \{v \in V \mid \|f - v\| = E_V(f)\}$$

A nonempty subset V of $C[a, b]$ is said to be an alternation system if there exists a mapping

$$d: V \to \mathbb{R}$$

such that $d(v) \geqslant 1$ for each $v \in V$ and if the following condition is fulfilled:

An element v in V is a best approximation of an f in $C[a, b]$ if and only if there exist points $t_0, t_1, \ldots, t_{d(v)}$ with $a \leqslant t_0 < t_1 < \cdots < t_{d(v)} \leqslant b$, and an $\eta \in \{-1, +1\}$ such that

$$\|f - v\| = \eta(-1)^k (f(t_k) - v(t_k)),$$

$k = 0, 1, \ldots, d(v)$.

The Main Result

THEOREM. *Let $V_1 \subset V_2$ be alternation systems of $C[a, b]$, $V_1, V_2 \subset C^1[a, b]$, $v_1 \in V_1$, and $v_2 \in V_2$. In order that there exists an element f in $C^1[a, b]$ such that $v_i \in P_{V_i}(f)$, $i = 1, 2$; it is necessary and sufficient that $v_1 - v_2$ is either zero or changes sign at atleast $d(v_1)$ points of $[a, b]$.*

For the proof we require the following lemmas.

LEMMA 1. *Let M_1 be a compact Hausdorff space in $\mathbb{R}^n$ and M_3 an open set containing M_1. Then there exists a function F defined on M_3 with compact support $M_2 \supset M_1$ with the following properties*:

(a) $F \in C^\infty[M_3]$,

(b) $F(t) = 1$ *for* $t \in M_1$,

(c) $0 \leqslant F(t) \leqslant 1$ *for* $t \in M_2 \backslash M_1$.

For the proof of the above lemma see de Rham [4, p. 4].

LEMMA 2. *Let U_1, U_{-1} be piecewise continuously differentiable functions in $C[a, b]$ and let*

$$M_1 := \{t_1, t_2, ..., t_k\},$$
$$M_{-1} := \{\tau_1, \tau_2, ..., \tau_j\}$$

be disjoint subsets of the open interval (a, b), with the following properties:

(i) $U_{-1}(t) < 0 < U_1(t)$, $\forall t \in [a, b]$.

(ii) *U_i is continuously differentiable at the points of M_i, $i = 1, -1$. Then there exists an $f \in C^1[a, b]$ such that*

(a) $f(t_\mu) = U_1(t_\mu)$, $\mu = 1, 2, ..., k$,

(b) $f(\tau_\nu) = U_{-1}(\tau_\nu)$, $\nu = 1, 2, ..., j$,

(c) $U_{-1}(t) \leqslant f(t) \leqslant U_1(t)$, $\forall t \in [a, b]$.

Proof. Set for $\mu = 1, 2, ..., k$ and λ_1, a positive real number,

$$G_\mu^1(t, \lambda_1) := U_1(t) - \lambda_1(t - t_\mu)^2,$$

and for $\nu = 1, 2, ..., j$ and λ_{-1}, a positive real number,

$$G_\nu^{-1}(t, \lambda_{-1}) := U_{-1}(t) + \lambda_{-1}(t - \tau_\nu)^2.$$

Then we have

$$G_\mu^1(t_\mu, \lambda_1) = U_1(t_\mu), \qquad \mu = 1, 2, ..., k, \tag{1}$$

$$G_\nu^{-1}(\tau_\nu, \lambda_{-1}) = U_{-1}(\tau_\nu), \qquad \nu = 1, 2, ..., j. \tag{2}$$

Further we have that G_μ^1 and G_ν^{-1} are continuously differentiable at t_μ and τ_ν respectively. By choosing λ_1 and λ_{-1} small enough, we have that in an open interval (c_μ, d_μ) of t_μ and an open interval (c_ν^{-1}, d_ν^{-1}) of τ_ν,

$$0 < G_\mu^1(t, \lambda_1) \leqslant U_1(t), \qquad \forall t \in (c_\mu, d_\mu), \tag{3}$$

$$0 > G_\nu^{-1}(t, \lambda_{-1}) \geqslant U_{-1}(t), \qquad \forall t \in (c_\nu^{-1}, d_\nu^{-1}), \tag{4}$$

and G_μ^1 restricted to (c_μ, d_μ) and G_ν^{-1} restricted to (c_ν^{-1}, d_ν^{-1}) are, respectively, in $C^1(c_\mu, d_\mu)$ and $C^1(c_\nu^{-1}, d_\nu^{-1})$. We can further assume, by taking these intervals sufficiently small, that these intervals are pairwise disjoint. Now consider the interval (c_μ, d_μ), $\mu = 1, 2,..., k$. Since $t_\mu \in (c_\mu, d_\mu)$, there exists a compact interval I_μ containing t_μ in its interior. Apply Lemma 1 with $M_1 = I_\mu$ and $M_3 = (c_\mu, d_\mu)$. Thus there exists a function F_μ and a compact set $\tilde{I}_\mu \supset I_\mu$ such that

($\bar{\text{a}}$) $F_\mu \in C^\infty(c_\mu, d_\mu)$,

($\bar{\text{b}}$) $F_\mu(t) = 1$ for $t \in I_\mu$,

($\bar{\text{c}}$) $0 \leqslant F_\mu(t) \leqslant 1$ for $t \in \tilde{I}_\mu \backslash I_\mu$.

We further define $F_\mu(t) = 0$ for $t \in [a, b] \backslash (c_\mu, d_\mu)$. Then it is clear that F_μ is in $C^\infty[a, b]$. Similarly we define for each ν, $\nu = 1, 2,..., j$; $F_\nu^{-1} \in C^\infty[a, b]$. We now set,

$$f(t) := \sum_{\mu=1}^{k} F_\mu(t)\, G_\mu^1(t, \lambda_1) + \sum_{\nu=1}^{j} F_\nu^{-1}(t)\, G_\nu^{-1}(t, \lambda_{-1})$$

for all t in $[a, b]$. We claim that $f \in C^1[a, b]$ and satisfies (a), (b), and (c) of the lemma. Notice that f is well defined on $[a, b]$. Since $F_\mu(t)\, G_\mu^1(t, \lambda_1) \in C^1[a, b]$ and $F_\nu^{-1}(t)\, G_\nu^{-1}(t, \lambda_{-1}) \in C^1[a, b]$, we have that $f \in C^1[a, b]$. If $t = t_\mu$, then from the definition of f it follows that

$$f(t_\mu) = F_\mu(t_\mu)\, G_\mu^1(t_\mu, \lambda_1)$$

since for all the indices $i \neq \mu$, $F_i(t) = 0$ on (c_μ, d_μ) and for all indices ν, $\nu = 1, 2,..., j$; $F_\nu^{-1}(t) = 0$ on (c_μ, d_μ). Since $F_\mu(t_\mu) = 1$,

$$f(t_\mu) = G_\mu^1(t_\mu, \lambda_1) = U_1(t_\mu)$$

This proves (a). Similarly, when $t = \tau_\nu$, we have $f(\tau_\nu) = U_{-1}(\tau_\nu)$. Finally, if $t \in (c_\mu, d_\mu)$, then, $0 \leqslant F_\mu(t) \leqslant 1$ and from (3), we have

$$U_{-1}(t) < 0 \leqslant f(t) = F_\mu(t)\, G_\mu^1(t, \lambda_1) \leqslant U_1(t).$$

Similarly, if $t \in (c_\nu^{-1}, d_\nu^{-1})$, we have $0 \leqslant F_\nu^{-1}(t) \leqslant 1$, and from (4) it follows that

$$U_1(t) > 0 \geqslant f(t) = F_\nu^{-1}(t)\, G_\nu^{-1}(t, \lambda_{-1}) \geqslant U_{-1}(t).$$

If $t \notin \{\bigcup_{\mu=1}^{k} (c_\mu, d_\mu)\} \cup \{\bigcup_{\nu=1}^{j} (c_\nu^{-1}, d_\nu^{-1})\}$, then clearly we have that

$$U_{-1}(t) < f(t) < U_1(t).$$

Combining these we have

$$U_{-1}(t) \leqslant f(t) \leqslant U_1(t) \qquad \text{for all} \quad t \in [a, b].$$

This completes the proof of Lemma 2.

Proof of the theorem. The proof is exactly the same as it is for the case when $V_1 \subset V_2 \subset C^2[a, b]$ (see Brosowski [2]). For the sake of completeness we give here the proof of the sufficiency part. The case $v_1 = v_2$ follows from Brosowski [1, Theorem 2.15, p. 62]. Let $v_1 \neq v_2$. Let $a_1, a_2, ..., a_{d(v_1)}$ be the zeros of $v_1 - v_2$ taken in their natural order. Put $a_0 = a$ and $a_{d(v_1)} + 1 = b$. Then we have

$$a = a_0 < a_1 < a_2 < \cdots < a_{d(v_1)} < a_{d(v_1)} + 1 = b.$$

Then there exists an $\eta \in \{-1, 1\}$ and for $i = 0, 1, 2, ..., d(v_1)$ points $t_{1,i}$ in (a_i, a_{i+1}) such that

$$\eta(-1)^i(v_2(t_{1,i}) - v_1(t_{1,i})) \geqslant \beta > 0$$

for $i = 0, 1, ..., d(v_1)$. Since $v_2 - v_1$ has at least one zero in (a, b), we can choose $d(v_2) + 1$ points $t_{2,0}, t_{2,1}, ..., t_{2,d(v_2)}$ such that

$$a \leqslant t_{2,0} < t_{2,1} < \cdots < t_{2,d(v_2)} \leqslant b$$

and

$$| v_2(t_{2,i}) - v_1(t_{2,i})| < \beta/2$$

for $i = 0, 1, ..., d(v_2)$. With a suitable real number $A > 0$, we set

$$U_1(t) := \min\{(v_1(t) + A + \beta), (v_2(t) + A + \beta/2)\} \qquad \text{(x)}$$

$$U_{-1}(t) := \max\{(v_1(t) - A - \beta), (v_2(t) - A - \beta/2)\} \qquad \text{(xx)}$$

Then we have, if A is sufficiently large, that

$$U_{-1}(t) < 0 < U_1(t) \qquad \text{for all} \quad t \in [a, b].$$

Notice that at a point $t_{1,i}$ with $\eta(-1)^i = +1$, we have

$$v_1(t_{1,i}) + A + \beta < v_2(t_{1,i}) + A + \beta/2.$$

Hence in a neighborhood of $t_{1,i}$ also we have the same inequality. Similarly, at a point $t_{1,s}$ with $\eta(-1)^s = -1$, we have that if $t \in$ a small neighborhood of $t_{1,s}$,

$$v_1(t) - A - \beta > v_2(t) - A - \beta/2.$$

This shows that U_1 is continuously differentiable at the points $t_{1,i}$ with $\eta(-1)^i = 1$ and U_{-1} is continuoulsy differentiable at the points $t_{1,s}$ with $\eta(-1)^s = -1$. Similarly we can show that at the points $t_{2,i}$, with i even, U_1 is continuously differentiable and U_{-1} is so at the points $t_{2,s}$ with s odd. Thus if we set

$$M_1 := \{t_{1,i} \text{ with } \eta(-1)^i = 1\} \cup \{t_{2,i} \text{ with } i \text{ even}\}$$
$$M_{-1} := \{t_{1,s} \text{ with } \eta(-1)^s = -1\} \cup \{t_{2,s} \text{ with } s \text{ odd}\},$$

we have that U_i is continuously differentiable on M_i, $i = 1, -1$. By Lemma 2 we have that there exists an $f \in C^1[a, b]$ such that

(i) $U_{-1}(t) \leqslant f(t) \leqslant U_1(t)$ for all $t \in [a, b]$,

(ii) $f(t) = U_1(t)$ on M_1,

(iii) $f(t) = U_{-1}(t)$ on M_{-1}.

Now from (x) and (xx) it follows that

$$-(A + \beta) \leqslant f(t) - v_1(t) \leqslant A + \beta$$
$$-(A + \beta/2) \leqslant f(t) - v_2(t) \leqslant A + \beta/2$$

for all $t \in [a, b]$. If $t = t_{1,i}$ with $\eta(-1)^i = 1$ then we have

$$f(t_{1,i}) = U_1(t_{1,i}) = v_1(t_{1,i}) + A + \beta$$

and so we have

$$f(t_{1,i}) - v_1(t_{1,i}) = A + \beta$$

while if $t = t_{1,s}$ with $\eta(-1)^s = -1$ we have

$$f(t_{1,s}) - v_1(t_{1,s}) = -A - \beta.$$

Similarly if $t = t_{2,i}$ with i even, we have

$$f(t_{2,i}) - v_2(t_{2,i}) = A + \beta/2$$

and if $t = t_{2,s}$ with s odd we have

$$f(t_{2,s}) - v_2(t_{2,s}) = -A - \beta/2.$$

Since V_1 and V_2 are alternation systems we conclude that

$$v_i \in P_{V_i}(f), \qquad i = 1, 2.$$

This completes the proof of the theorem.

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